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LETTER TO THE EDITOR

Soliton interactions (for the Korteweg–deVries equation): a new perspective

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Abstract. It can be shown that the N-soliton solution of the Korteweg-deVries equation can be decomposed into N separate solitons (cf Gardner *et al*, Calogero and Degasperis and Yoneyama). However, it is not immediately clear from the form of their solutions how the separate solitons relate directly to the single soliton solution. Here the two-soliton case is considered and a decomposition is sought which can be clearly related to the single soliton solution. Although it appears that there is a family of such decompositions it is shown that only one of these is correct. Although this decomposition is equivalent to the decomposition given previously by Gardner *et al*, Calogero and Degasperis and Yoneyama, the form given here is different. It is suggested that the form of solution produced here is a more appropriate representation of the solution since it is clear how it relates directly to the single soliton solution and it is easy, through this form, to analyse the interaction of the two solitons.

The well known solitary wave or soliton solution of the Korteweg-deVries equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

is

$$u = 2a^2\operatorname{sech}^2(\theta)$$

where

$$\theta = ax - 4a^3t \tag{2}$$

(cf Dodd *et al* (1982) p 11, where the equation and solution are written in slightly different form).

The solution (2) yields the conservation property that

$$\int_{-\infty}^{\infty} u \, dx = 4a. \tag{3a}$$

We define the path of the soliton as the function $x(t)$ such that

$$\int_{-\infty}^{x(t)} u \, dx = \int_{x(t)}^{\infty} u \, dx = 2a \tag{3b}$$

so that at any point along the path equal portions of the soliton are on either side of the point $x(t)$. It is clear, for the function u as given by expression (2), that ($3b$) is satisfied by that $x(t)$ for which $\theta = 0$. Hence the path is given by putting the argument of the sech^2 function in (2) equal to zero.

Hirota (1971) showed how to find multi-soliton solutions of the KdV equation and following his method the two-soliton solution is

$$u = 2(ff_{xx} - f_x^2)/f^2 \tag{4a}$$

where

$$f = 1 + \exp(2\theta_1) + \exp(2\theta_2) + A\exp(2\theta_1 + 2\theta_2) \tag{4b}$$

$$\theta_1 = a_1 x - 4a_1^3 t \qquad \theta_2 = a_2 x - 4a_2^3 t$$

$$A = [(a_1 - a_2)/(a_1 + a_2)]^2.$$

Consider a decomposition of the solution u, i.e.

$$u = u_1 + u_2. \tag{5}$$

Although there are infinitely many ways to choose u_1 and u_2 one looks for choices such that both u_1 and u_2 have a form similar to expression (2). A possible choice of this type is

$$u_1 = 2a_1^2 H(\theta_2)\,\mathrm{sech}^2[\theta_1 + G(\theta_2)] \tag{6a}$$

$$u_2 = 2a_2^2 H(\theta_1)\,\mathrm{sech}^2[\theta_2 + G(\theta_1)] \tag{6b}$$

where the expressions (6a) and (6b) for u_1 and u_2 are more general forms of expression (2). These generalisations involve a variable amplitude and a variable phase shift.

Furthermore the variation of u_1 from a single soliton is entirely due to θ_2, the characteristic of u_2. Likewise the variation of u_2 from a single soliton is entirely due to θ_1, the characteristic of u_1. Hence the deviation of u_1 from a single soliton behaviour is due to the presence of u_2 and vice versa for u_2. One would expect that when u_1 and u_2 are well separated these effects would diminish, i.e. $H(\theta_1)$ and $H(\theta_2)$ would tend to unity and $G(\theta_1)$ and $G(\theta_2)$ would tend to constant values.

It is possible to find a family of such function pairs (cf the appendix) with the phase shift and amplitude factors, in (6a) and (6b), given by

$$H(\theta_1) = \frac{1 + B_1\exp(2\theta_1) + A\exp(4\theta_1)}{1 + (1+A)\exp(2\theta_1) + A\exp(4\theta_1)} \tag{7}$$

$$H(\theta_2) = \frac{1 + B_2\exp(2\theta_2) + A\exp(4\theta_2)}{1 + (1+A)\exp(2\theta_2) + A\exp(4\theta_2)}$$

$$G(\theta_1) = \tfrac{1}{2}\ln\left(\frac{1 + A\exp(2\theta_1)}{1 + \exp(2\theta_1)}\right)$$

$$G(\theta_2) = \tfrac{1}{2}\ln\left(\frac{1 + A\exp(2\theta_2)}{1 + \exp(2\theta_2)}\right)$$

where B_1 and B_2 are arbitrary constants obeying the constraint

$$a_1^2 B_2 + a_2^2 B_1 = 2(a_1 - a_2)^2. \tag{8}$$

From the range of possible choices for B_1 and B_2 the most obvious choice appears to be

$$B_1 = \left(\frac{a_1 - a_2}{a_2}\right)^2 \qquad B_2 = \left(\frac{a_1 - a_2}{a_1}\right)^2. \tag{9}$$

However, an additional constraint on the values of B_1 and B_2 is that the expressions u_1 and u_2 given in ($6a$) and ($6b$) should satisfy the conservation property ($3a$), i.e.

$$\int_{-\infty}^{\infty} u_1 \, \mathrm{d}x = 4a_1 \tag{10a}$$

$$\int_{-\infty}^{\infty} u_2 \, \mathrm{d}x = 4a_2. \tag{10b}$$

We therefore need to test whether B_1 and B_2 as given by (9) lead to expressions for u_1 and u_2 such that ($10a$) and ($10b$) are satisfied. To do this we use expressions (A1) and ($4b$) to rewrite u_1 given by ($6a$), as

$$u_1 = \frac{8a_1^2 \exp(2\theta_1)[1 + B_2 \exp(2\theta_2) + A \exp(4\theta_2)]}{[1 + \exp(2\theta_1) + \exp(2\theta_2) + A \exp(2\theta_1 + 2\theta_2)]^2}. \tag{11}$$

Expression (11) can be rewritten as

$$\begin{aligned} u_1 = {} & \frac{8a_1^2 \exp(2\theta_1)[1 + 2A^{1/2} \exp(2\theta_2) + A \exp(4\theta_2)]}{[1 + \exp(2\theta_1) + \exp(2\theta_2) + A \exp(2\theta_1 + 2\theta_2)]^2} \\ & + \frac{8a_1^2 \exp(2\theta_1 + 2\theta_2)[B_2 - 2A^{1/2}]}{[1 + \exp(2\theta_1) + \exp(2\theta_2) + A \exp(2\theta_1 + 2\theta_2)]^2}. \end{aligned} \tag{12}$$

Expression (12) may also be rewritten as

$$\begin{aligned} u_1 = {} & 4a_1 \frac{\delta}{\delta x} \left(\frac{\exp(2\theta_1) + A \exp(2\theta_1 + 2\theta_2)}{1 + \exp(2\theta_1) + \exp(2\theta_2) + A \exp(2\theta_1 + 2\theta_2)} \right) \\ & + 8a_1^2 F(\theta_1, \theta_2)[B_2 - 2A^{1/2}] \end{aligned} \tag{13}$$

where

$$F(\theta_1, \theta_2) = \exp(2\theta_1 + 2\theta_2)/[1 + \exp(2\theta_1) + \exp(2\theta_2) + A \exp(2\theta_1 + 2\theta_2)]^2. \tag{14}$$

Note that $F(\theta_1, \theta_2) > 0$ in all parts of the xt plane. Then

$$\int_{-\infty}^{\infty} u_1 \, \mathrm{d}x = 4a_1 + 8a_1^2 (B_2 - 2A^{1/2}) \int_{-\infty}^{\infty} F(\theta_1, \theta_2) \, \mathrm{d}x \tag{15}$$

so that expression ($10a$) is satisfied only if

$$B_2 = 2A^{1/2}. \tag{16}$$

Similarly, expression ($10b$) is satisfied only if

$$B_1 = -2A^{1/2}. \tag{17}$$

One therefore sees that the, apparently obvious, choice (9) for B_1 and B_2 is not possible since the conservation properties ($10a$) and ($10b$) are not satisfied in this case. It therefore transpires that the only possible choices for B_1 and B_2 are those given by expressions (16) and (17).

The solution u as given by expressions (5)-(7) with B_1, B_2 given by expression (16), (17) is identical to the solution for this problem previously derived by Gardner *et al* (1974) through the inverse scattering method (cf also Calogero and Degasperis (1982) § 3.2.1) and separately by Yoneyama (1984).

However, the form of the solution given here is different to the forms of previous solutions and preferable in the sense that it is clear how the solution (as given by expressions (5) and (6a) and (b)) relates to the single soliton solution (2). It is also shown subsequently that it is possible to easily analyse the interaction of the two solitons using the form of the solution given here.

For definiteness we consider the case $a_1 > a_2 > 0$. Note that expression (5) for u is the sum of two terms where each term has the form of a modified soliton in the sense that the first term represents a soliton in the variable θ_1 when θ_2 is constant while the second term represents a soliton in the variable θ_2 when θ_1 is constant. It is therefore to be expected that (5) should yield the sum of two solitons as $t \to \pm\infty$ by considering the asymptotic behaviour of the first term of (5) for *any fixed* θ_1 (as $t \to \pm\infty$) and the asymptotic behaviour of the second term of (5) for *any fixed* θ_2 (as $t \to \pm\infty$). The limits are: for *fixed* θ_1, $\theta_2 \to -\infty$ as $t \to -\infty$ and $\theta_2 \to \infty$ as $t \to \infty$ (since $a_1 > a_2$); for *fixed* θ_2, $\theta_1 \to \infty$ as $t \to -\infty$ and $\theta_1 \to -\infty$ as $t \to \infty$ (since $a_1 > a_2$).

The asymptotic behaviour of u, as given by (5), under the above limiting procedure is then

$$u_1(t \to -\infty) \sim 2a_1^2 \operatorname{sech}^2(\theta_1) \tag{18a}$$

$$u_2(t \to -\infty) \sim 2a_2^2 \operatorname{sech}^2[\theta_2 + \tfrac{1}{2}\ln A] \tag{18b}$$

$$u_1(t \to \infty) \sim 2a_1^2 \operatorname{sech}^2[\theta_1 + \tfrac{1}{2}\ln A] \tag{19a}$$

$$u_2(t \to \infty) \sim 2a_2^2 \operatorname{sech}^2(\theta_2). \tag{19b}$$

Expressions (18a)-(19b) summarise the behaviour of the two solitons. As $t \to -\infty$, u_1, with larger magnitude ($2a_1^2$) and larger phase speed ($4a_1^2$) lies on the x axis at $x = 4a_1^2 t$ which is further to the left than u_2, with magnitude ($2a_2^2$) and phase speed ($4a_2^2$) which lies at $x = 4a_2^2 t - \ln A/2a_2$ on the x axis. As $t \to +\infty$, u_1 lies at $x = 4a_1^2 t - \ln A/2a_1$ on the x axis which is to the right of u_2 which is at $x = 4a_2^2 t$ on the x axis. The two solitons, u_1 and u_2, have the same form and speed as $t \to -\infty$ and as $t \to \infty$ as they would have if they were single solitons. The only effect on them over time is that u_1 has undergone a phase shift of $-\frac{1}{2}\ln A$ forward along the θ_1 axis while u_2 has undergone a phase shift of $-\frac{1}{2}\ln A$ backward along the θ_2 axis.

By direct analogy with (3b) the path of soliton u_1 is given from (6a) by putting the argument of the sech^2 function equal to zero, i.e.

$$\theta_1 = -\tfrac{1}{2}\ln[(1 + A\exp(2\theta_2))/(1 + \exp(2\theta_2))]$$

which may be rewritten as

$$\exp(2\theta_1) = [1 + \exp(2\theta_2)]/[1 + A\exp(2\theta_2)]. \tag{20}$$

In expression (20) t should be considered as the independent variable and x the dependent variable, i.e. $x = x_1(t)$ where $x_1(t)$ is the solution of (20). The function $x_1(t)$ also satisfies the analogous expression to (3b), i.e.

$$\int_{-\infty}^{x_1(t)} u_1\,dx = \int_{x_1(t)}^{\infty} u_1\,dx = 2a_1. \tag{21}$$

In a similar way the path of the soliton u_2 is given by

$$\exp(2\theta_2) = [1 + \exp(2\theta_1)]/[1 + A\exp(2\theta_1)]. \tag{22}$$

If the interaction point of the solitons is defined as the point of intersection of their paths, this point is given (from (20) and (22)) by

$$\theta_1 = \theta_2 = -\tfrac{1}{4}\ln A. \tag{23}$$

When expression (23) is solved for x and t the interaction point is found to be

$$x = -\tfrac{1}{4}(a_1^2 + a_1 a_2 + a_2^2)\ln A/[a_1 a_2(a_1 + a_2)] \qquad t = -\tfrac{1}{16}\ln A/[a_1 a_2(a_1 + a_2)]. \tag{24}$$

The paths of the solitons as given by expressions (20) and (22) are shown in figure 1. Note the paths are symmetric about the interaction point given by expression (23). Substituting expression (23) into expressions (5) and (6a) and (6b) gives $u = 2(a_1^2 - a_2^2)$ so that at the interaction point the amplitude of the 'double' soliton is the difference of the asymptotic amplitudes of the two separate solitons. Note that two solitons of nearly equal phase speeds (and hence magnitudes) almost annihilate each other at the interaction point. Also the amplitude of the slower soliton is reduced to zero at the interaction point. As is clearly shown in figure 1 the slower soliton moves backwards while the faster soliton accelerates forwards near the interaction point.

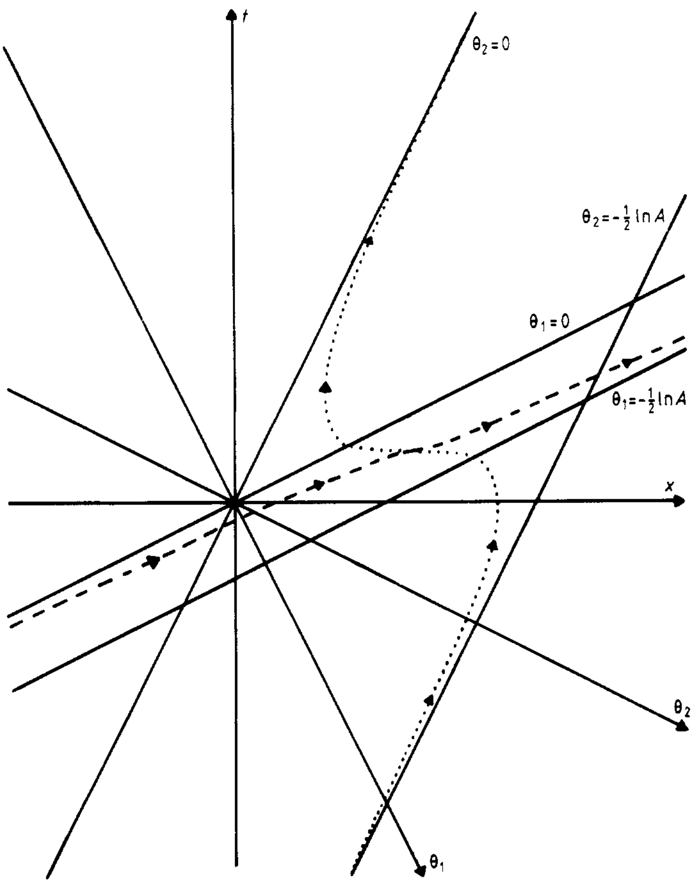

Figure 1. The paths of the solitons are shown in the xt plane. The path of one soliton is the broken curve and that of the other is the dotted curve. Also shown are the θ_1 and θ_2 axes and the asymptotes of the paths. The arrows indicate the directions of motion along the paths.

It is instructive to consider the following extreme cases: (1) $a_2 \to 0$, i.e. $a_2 \ll a_1$ and (2) $a_2 \to a_1$.

Case (1) is given by $a_2 = \varepsilon a_1$ where $0 < \varepsilon \ll 1$ so that the interaction point (x, t) as given by expression (24) becomes $x \to 1/a_1$ and $t \to \frac{1}{4}/a_1^3$ or alternatively on using expression (23) the interaction point $\theta_1 = \theta_2 = -\frac{1}{4}\ln A$ becomes $\theta_1 = \theta_2 = 2\varepsilon$ which gives a point near the origin in the (θ_1, θ_2) plane.

Case (2) is given by $a_2 = a_1(1-\varepsilon)$ where $0 < \varepsilon \ll 1$ so that the interaction point (x, t) as given by expression (24) becomes $x \to -3\ln(\frac{1}{2}\varepsilon)/4a_1$ and $t \to -\ln(\frac{1}{2}\varepsilon)/16a_1^3$ which give *very large positive* values for both x and t. Alternatively on using expression (23) the interaction point $\theta_1 = \theta_2 = -\frac{1}{4}\ln A$ becomes $\theta_1 = \theta_2 = -\frac{1}{2}\ln(\frac{1}{2}\varepsilon)$ which also gives very large positive values for θ_1 and θ_2.

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Appendix

Expression (4) for u can be rearranged as follows: expression (4*b*) gives

$$ff_{xx} - f_x^2 = 4[a_1^2\exp(2\theta_1) + a_2^2\exp(2\theta_2) + 2(a_1-a_2)^2\exp(2\theta_1+2\theta_2) + a_2^2A\exp(4\theta_1+2\theta_2) + a_1^2A\exp(2\theta_1+4\theta_2)]$$

which can be rewritten as

$$ff_{xx} - f_x^2 = 4a_1^2 p(\theta_2)\exp(2\theta_1) + 4a_2^2 p(\theta_1)\exp(2\theta_2) \tag{A1}$$

where

$$p(\theta_2) = 1 + B_2\exp(2\theta_2) + A\exp(4\theta_2)$$
$$p(\theta_1) = 1 + B_1\exp(2\theta_1) + A\exp(4\theta_1)$$
$$a_2^2B_1 + a_1^2B_2 = 2(a_1-a_2)^2. \tag{A2}$$

Expression (4*b*) also gives

$$\begin{aligned} f^2 &= [1+\exp(2\theta_1)]^2 + 2[1+\exp(2\theta_1)][\exp(2\theta_2) + A\exp(2\theta_1+2\theta_2)] \\ &\quad + [\exp(2\theta_2) + A\exp(2\theta_1+2\theta_2)]^2 \\ &= [1+\exp(2\theta_1)][1+A\exp(2\theta_1)]\exp(2\theta_2) \\ &\quad \times\left[\left(\frac{1+\exp(2\theta_1)}{1+A\exp(2\theta_1)}\right)\exp(-2\theta_2) + 2 + \left(\frac{1+A\exp(2\theta_1)}{1+\exp(2\theta_1)}\right)\exp(2\theta_2)\right] \\ &= 4q(\theta_1)\exp(2\theta_2)\cosh^2[\theta_2 + G(\theta_1)] \end{aligned} \tag{A3}$$

where

$$q(\theta_1) = [1+\exp(2\theta_1)][1+A\exp(2\theta_1)] = 1 + (1+A)\exp(2\theta_1) + A\exp(4\theta_1)$$

and

$$G(\theta_1) = \tfrac{1}{2}\ln[(1+A\exp(2\theta_1))/(1+\exp(2\theta_1))].$$

Since expression (4b) for f is symmetric in θ_1 and θ_2 therefore expression (A3) for f^2 can be rewritten, by interchanging θ_1 and θ_2, to obtain

$$f^2 = 4q(\theta_2)\exp(2\theta_1)\cosh^2[\theta_1 + G(\theta_2)]. \tag{A4}$$

Then substituting expressions (A1) and (A3) or (A4) as appropriate into expression (4a) for u gives

$$u = \frac{2a_1^2 p(\theta_2)\exp(2\theta_1)}{q(\theta_2)\exp(2\theta_1)\cosh^2[\theta_1 + G(\theta_2)]} + \frac{2a_2^2 p(\theta_1)\exp(2\theta_2)}{q(\theta_1)\exp(2\theta_2)\cosh^2[\theta_2 + G(\theta_1)]} \tag{A5}$$

which is expression (5), on using (6a) and (6b), for u as given earlier.

Note that since B_1 and B_2 are not, as yet, fixed in (A2) there is a family of solutions of type (A5).

References

Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* (Amsterdam: North-Holland)

Dodd R K, Eilbeck J C, Gibbon J D and Morris H C 1982 *Solitons and Nonlinear Wave Equations* (New York: Academic)

Gardner C S, Greene J M, Kruskal M D and Miura R M 1974 *Commun. Pure Appl. Math.* **27** 97-133

Hirota R 1971 *Phys. Rev. Lett.* **27** 1192-4

Yoneyama T 1984 *Prog. Theor. Phys.* **72** 1081-8